

HADAMARD KASR TARTIBLI HOSILALI SUBDIFFUZIYA TENGLAMASI

Fayziev Yusuf

O'zbekiston Milliy universiteti Differensial tenglama va matematik fizika

tenglamasi kafedrasida dotsenti

fayziev.yusuf@mail.ru

Sulaymonov Ilyosxo'ja

O'zbekiston Milliy universiteti 4-bosqich talabasi

sulaymonovilyosxoja@gmail.com

Annotatsiya. Ushbu maqolada Hadamard ma'nosidagi kasr tartibli subdiffuziya tenglamasi uchun Koshi masalasi o'rganilgan.

Kalit so'zlar: Hadamard kasr tartibli hosilasi, Mittag-Leffler funksiyasi, operator.

Аннотация. В данной работе исследуется задача Коши для дробного уравнения субдиффузии, означающего Адамара.

Ключевые слова: дробная производная Адамара, функция Миттаг-Леффлера, оператор.

Abstract. This article studied the Cauchy problem for the fractional subdiffusion equation with Hadamard derivative.

Keywords: Hadamard fractional derivative, Mittag-Leffler function, operator.

So'nggi yillarda fizika, biologiya va texnika fanlarida kasr tartibli tenglamalarining qo'llanilishi tufayli ularni o'rganishga bo'lgan qiziqish katta sezilarli darajada ortdi. Kasr tartibli tenglamalar ko'plab matematiklar tomonidan o'rganilgan. Bu haqda ko'proq ma'lumotlarni [1] - [7] ishlarda topish mumkin.

Faraz qilaylik, H separabel Hilbert fazosi bo'lsin. H da skalyar ko'paytmani (\cdot, \cdot) orqali, normani esa $\|\cdot\|$ kabi belgilaymiz. $A: H \rightarrow H$ chegaralanmagan, o'z-o'ziga qo'shma, musbat aniqlangan ixtiyoriy operator bo'lsin. Shu bilan birga A operatorning teskarisi mavjud va kompakt operator bo'lsin. Aytaylik, H fazodagi $\{v_k\}$ to'la

ortanormal sistema A operatorning $\{\lambda_k\}$ sanoqli musbat xos sonlariga mos xos funksiyalari bo'lsin. Qulaylik uchun $\{\lambda_k\}$ xos sonlar ketma – ketligi limit nuqtaga ega emas va qayta nomerlash orqali $0 < \lambda_1 \leq \lambda_2 \dots \rightarrow +\infty$ ko'rinishda yozib olamiz. $C((a,b);H)$ orqali qiymatlari H da yotuvchi $t \in (a,b)$ oraliqda uzluksiz $u(t)$ funksiyalar to'plamini belgilaylik.

Aytaylik, $\delta = x \frac{d}{dx}$ va $n-1 < \alpha < n$ bo'lsin. Hadamardning α - kasr tartibli

hosilasi deb quyidagiga ifodaga aytiladi:

$$D_x^\alpha y(x) := \delta^n (I^{n-\alpha} y(x)) = \left(x \frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \log\left(\frac{x}{\xi}\right)^{n-\alpha-1} \frac{y(\xi)d\xi}{\xi}, \quad (a < x < b). \tag{1.1}$$

Aytaylik, $0 < \alpha < 1$ bo'lsin. Quyidagi Koshi masalasini qaraylik:

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f, & t > 1 \\ \lim_{t \rightarrow 1+0} I_t^{\alpha-1} u(t) = \varphi, \end{cases} \tag{1.2}$$

bu yerda $\varphi, f \in H$.

Ta'rif. Ushbu $D_t^\alpha u(t), Au(t) \in C((1,\infty);H)$ va (1.2) shartni qanoatlantiruvchi $(\ln t)^{1-\alpha} \cdot u(t) \in C((1,\infty);H)$ funksiyaga (1.2) masalaning yechimi deyiladi.

Endi biz (1.2) to'g'ri masalaning yechimi mavjud va yagonaligini ko'rsatamiz.

Buning uchun avval Mittag-Leffler funksiyasining keyinchalik bizga kerak bo'ladigan ba'zi xossalari eslatib o'tamiz. $E_{\rho,\mu}(z)$ bilan ikki parametrlil Mittag-Leffler funksiyasini belgilaymiz:

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}. \tag{2.1}$$

Bizga Mittag-Leffler funksiyasi uchun ba'zi baholashlar kerak. Yetarlicha katta t uchun asimptotik baho mavjud ([1], p. 75, [3], p. 13)

$$E_{\rho, \rho+1}(-t) = \frac{1}{t} \left(1 + O\left(\frac{1}{t}\right) \right), \quad (2.2)$$

va ixtiyoriy kompleks son μ uchun

$$0 < |E_{\rho, \mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0. \quad (2.3)$$

Teorema 1. Aytaylik $0 < \alpha < 1$ va $\varphi, f \in H$. U holda (1.2) masala yagona yechimga ega va u quyidagi ko‘rinishda bo‘ladi:

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k (\ln t)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k (\ln t)^{\alpha}) + f_k (\ln t)^{\alpha} E_{\alpha, \alpha+1}(-\lambda_k (\ln t)^{\alpha}) \right] v_k. \quad (2.4)$$

bu yerda φ_k va f_k lar mos ravishda φ va f funksiyalarning Furye koeffitsiyentlari, hamda $E_{\rho, \mu}(z)$ - Mittag – Liffler funksiyasi.

Isbot. Yechimning mavjudligi. Biz bu bo‘limda (1.2) masalaning yechimi mavjud va yagonaligini ko‘rsatamiz. (1.2) masalani yechish uchun matematik fizika masalalarini yechishda keng qo‘llaniladigan usullardan biri Furye usulidan foydalanamiz, ya’ni yechimni quyidagi ko‘rinishda izlaymiz:

$$u(x, t) = \sum_{k=1}^{\infty} T_k(t) v_k. \quad (2.5)$$

(2.5) qatorni (1.2) tenglamaga olib borib quyamiz va quyidagi tenglikni hosil qilamiz:

$$\sum_{k=1}^{\infty} D_t^{\alpha} T_k(t) v_k + \sum_{k=1}^{\infty} T_k(t) A v_k = f. \quad (2.6)$$

$A v_k = \lambda_k v_k$ tenglikni hisobga olib, hamda f ni $\{v_k\}$ to‘la sistema bo‘yicha qatorga yoyib, quyidagi

$$\sum_{k=1}^{\infty} (D_t^{\alpha} T_k(t) + \lambda_k T_k(t)) v_k = \sum_{k=1}^{\infty} f_k v_k \quad (2.7)$$

tenglikka ega bo‘lamiz. Bundan esa quyidagi Koshi masalasiga kelamiz:

$$\begin{cases} D_t^\alpha T_k(t) + \lambda_k T_k(t) = f_k, \\ \lim_{t \rightarrow 1+0} I_T^{\alpha-1} T_k(t) = \varphi_k. \end{cases} \quad (2.8)$$

Bu masalani yechimi quyidagi ko‘rinishda bo‘ladi (qarang [2]):

$$T_k(t) = \varphi_k \left(\ln \frac{x}{t} \right)^{\alpha-1} E_{\alpha,\alpha} \left(-\lambda_k \left(\ln \frac{x}{t} \right)^\alpha \right) + f_k \int_1^x \left(\ln \frac{x}{t} \right)^\alpha E_{\alpha,\alpha+1} \left(-\lambda_k \left(\ln \frac{x}{t} \right)^\alpha \right) \frac{dt}{t}$$

bir nechta shakl almashtirishlardan so‘ng (2.8) masalaning yechimi quyidagicha bo‘ladi

$$T_k(t) = \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha} (-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1} (-\lambda_k (\ln t)^\alpha). \quad (2.9)$$

(2.9) yechimni (2.4) ga olib borib qo‘ysak quyidagiga kelamiz:

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha} (-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1} (-\lambda_k (\ln t)^\alpha) \right] v_k \quad (2.10)$$

(2.10) qator bilan berilgan $u(t)$ funksiya (1.2) masalaning formal yechimi ekanligi kelib chiqdi. Endi ushbu yechimning haqiqiy yechim ekanligini ko‘rsatish uchun ushbu qatorga D_t^α va A operatorlarni hadma-had qo‘llashimiz kerak bo‘ladi. Shuning uchun (2.10) qatorning tekis va absolyut yaqinlashuvchi ekanligini ko‘rsatish kerak bo‘ladi.

Buning uchun (2.10) qatorning qisman yig‘indisi

$$S_n(t) = \sum_{k=1}^n \left[\varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha} (-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1} (-\lambda_k (\ln t)^\alpha) \right] v_k \quad (2.11)$$

ni olamiz. Ushbu qisman yig‘indini baholaylik:

$$\|S_n(t)\|^2 = \left| \sum_{k=1}^n \left[\varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha} (-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1} (-\lambda_k (\ln t)^\alpha) \right] v_k \right|^2$$

Parseval tengligiga ko‘ra, ya’ni

$$\|S_n(t)\|^2 = \sum_{k=1}^n \left| \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right|^2$$

tenglikka ega bo'lamiz. Endi ushbu ifodani baholaylik

$$\begin{aligned} \|S_n(t)\|^2 &\leq C \sum_{k=1}^n \left| \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) \right|^2 + \\ &+ C \sum_{k=1}^n \left| f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right|^2 = S_n^1 + S_n^2, \end{aligned}$$

bu yerda

$$\begin{aligned} S_n^1 &= C \sum_{k=1}^n \left| \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) \right|^2, \\ S_n^2 &= C \sum_{k=1}^n \left| f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right|^2. \end{aligned}$$

Endi (2.3) bahodan foydalanib, yuqoridagi tenglikning o'ng tomonini har birini baholaylik:

$$\begin{aligned} S_n^1 &= C \sum_{k=1}^n \left| \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) \right|^2 \leq C \sum_{k=1}^n |\varphi_k|^2 (\ln t)^{2\alpha-2} \left| \frac{1}{1 + \lambda_k (\ln t)^\alpha} \right|^2 \leq \\ &\leq C \sum_{k=1}^n |\varphi_k|^2 (\ln t)^{2\alpha-2} \frac{1}{\lambda_k^2 (\ln t)^{2\alpha}} \leq \frac{C}{\lambda_1^2 (\ln t)^2} \sum_{k=1}^n |\varphi_k|^2. \end{aligned}$$

va

$$\begin{aligned} S_n^2 &= C \sum_{k=1}^n \left| f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right|^2 \leq C \sum_{k=1}^n |f_k|^2 (\ln t)^{2\alpha} \left| \frac{1}{1 + \lambda_k (\ln t)^\alpha} \right|^2 \leq \\ &\leq C \sum_{k=1}^n |f_k|^2 (\ln t)^{2\alpha} \frac{1}{\lambda_k^2 (\ln t)^{2\alpha}} \leq \frac{C}{\lambda_1^2} \sum_{k=1}^n |f_k|^2. \end{aligned}$$

Oxirgi uchta tengsizlikdan ko'rinib turibdiki, agar $\varphi, f \in H$ bo'lsa, u holda (2.10) qator yaqinlashuvchi bo'ladi. Bundan $(\ln t)^{1-\alpha} \cdot u(t) \in C((1, \infty); H)$ ekanligi kelib

chiqadi. Endi $Au(t)$ ni baholaylik, $S_n(t)$ yig'indi (2.10) qatorning qisman yig'indisi bo'lgani uchun

$$AS_n(t) = \sum_{k=1}^n \left(\varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right) \lambda_k v_k \tag{2.13}$$

tenglik o'rinli bo'ladi. Parseval tengligiga ko'ra

$$\begin{aligned} \|AS_n(t)\|^2 &= \left| \sum_{k=1}^n \left(\varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right) v_k \lambda_k \right|^2 = \\ &= \sum_{k=1}^n \lambda_k^2 \left| \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) + f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right|^2 \end{aligned}$$

bo'ladi. U holda biz quyidagi tengsizlikka ega bo'lamiz:

$$\begin{aligned} \|AS_n(t)\|^2 &\leq C \sum_{k=1}^n \lambda_k^2 \left| \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) \right|^2 + \\ &+ C \sum_{k=1}^n \lambda_k^2 \left| f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right|^2 = AS_n^1 + AS_n^2. \end{aligned}$$

Agar

$$\begin{aligned} AS_n^1 &= C \sum_{k=1}^n \lambda_k^2 \left| \varphi_k (\ln t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (\ln t)^\alpha) \right|^2, \\ AS_n^2 &= C \sum_{k=1}^n \lambda_k^2 \left| f_k (\ln t)^\alpha E_{\alpha,\alpha+1}(-\lambda_k (\ln t)^\alpha) \right|^2 \end{aligned}$$

kabi belgilashlar kiritsak, u holda quyidagi tengsizlikka ega bo'lamiz:

$$\|AS_n(t)\|^2 \leq AS_n^1 + AS_n^2.$$

Endi (2.3) bahodan foydalanib, yuqoridagi tengsizlikning o'ng tomonini har birini baholaylik:

$$AS_n^1 \leq C \sum_{k=1}^n \lambda_k^2 |\varphi_k|^2 |\ln t|^{2\alpha-2} \left| \frac{1}{1 + \lambda_k (\ln t)^\alpha} \right|^2$$

$$\leq C \sum_{k=1}^n \lambda_k^2 |\varphi_k|^2 (\ln t)^{2\alpha-2} \frac{1}{\lambda_k^2 (\ln t)^{2\alpha}} = \frac{C}{(\ln t)^2} \sum_{k=1}^n |\varphi_k|^2.$$

va

$$AS_n^2 = C \sum_{k=1}^n \lambda_k^2 |f_k (\ln t)^\alpha E_{\alpha, \alpha+1}(-\lambda_k (\ln t)^\alpha)|^2 \leq C \sum_{k=1}^n \lambda_k^2 |f_k|^2 (\ln t)^{2\alpha} \left| \frac{1}{1 + \lambda_k (\ln t)^\alpha} \right|^2 \leq C \sum_{k=1}^n \lambda_k^2 |f_k|^2 (\ln t)^{2\alpha} \frac{1}{\lambda_k^2 (\ln t)^{2\alpha}} = C \sum_{k=1}^n |f_k|^2.$$

Demak, agar $\varphi, f \in H$ bo'lsa, u holda $Au(t) \in C((1, \infty); H)$ bo'ladi.

Endi masalaning yechimi yagona ekanligini isbotlaymiz. Teskarisini faraz qilamiz, ya'ni masalaning yechimi ikkita $u_1(t)$ va $u_2(t)$ bo'lsin. $u_1(t) \neq u_2(t)$ va ularning ayirmasi $u_1(t) - u_2(t) = u(t)$ bo'lsin.

U holda berilgan tenglamani quyidagi ko'rinishlarda yozib olamiz:

$$\begin{cases} D_t^\alpha u_1(t) + Au_1(t) = f \\ \lim_{t \rightarrow 1+0} I_t^{\alpha-1} u_1(t) = \varphi \end{cases} \quad (2.14)$$

va

$$\begin{cases} D_t^\alpha u_2(t) + Au_2(t) = f \\ \lim_{t \rightarrow 1+0} I_t^{\alpha-1} u_2(t) = \varphi \end{cases} \quad (2.15)$$

Endi (2.14) tenglamadan (2.15) tenglamani hadma had ayirib yuboramiz.

$$\begin{cases} D_t^\alpha u(t) + Au(t) = 0 \\ \lim_{t \rightarrow 1+0} I_t^{\alpha-1} u(t) = 0 \end{cases} \quad (2.16)$$

masalaga kelamiz.

Yagonaligi. Aytaylik $u(t)$ (2.16) masalaning ixtiyoriy yechimi bo'lsin. U holda $u_k(t) = (u(t), v_k)$ deb izlaymiz. U holda yuqoridagi tenglik va A operatorning o'z – o'ziga qo'shmaligidan quyidagi tenglamani hosil qilamiz.

$$\begin{aligned} D_t^\alpha u_k(t) &= (D_t^\alpha u(t), v_k) = -(Au(t), v_k) = -(u(t), Av_k) = \\ &= -(u(t), \lambda_k v_k) = -\lambda_k (u(t), v_k) = -\lambda_k u_k(t), \quad t > 1 \end{aligned} \quad (2.17)$$

Agar Koshi shartidan foydalansak, unda quyidagi masalaga ega bo'lamiz.

$$D_t^\alpha u_k(t) + \lambda_k u_k(t) = 0, \quad t > 1; \quad u(t) = 0$$

Bundan ko'rinib turibdiki $u_k(t) \equiv 0$ ekanligi kelib chiqadi. Bundan esa $\{v_k\}$ xos funksiyalarning to'laligidan $u(t) \equiv 0$ ekanligi kelib chiqadi. Teorema to'la isbotlandi.

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