

CENTRAL LIMIT THEOREM FOR M-DEPENDENT RANDOM VARIABLES

Muxtorov Ibrohim G‘aybulla o‘g‘li

National University of Uzbekistan, Tashkent

ibrohimmuxtorov199702@gmail.com

Abstract. In this article, I proved central limit theorem for m-dependent random variables.

Keywords: Central limit theorem, m-dependent random variables

Annotatsiya. Bu maqolada m-bog‘liqli tasodifiy miqdorlar uchun markaziy limit teorema isbotlaganman.

Kalit so‘zlar: Markaziy limit teorema, m-bog‘liqli tasodifiy miqdorlar.

Аннотация. В этой статье я доказал центральную предельную теорему для m- зависимых случайных величин.

Ключевые слова: Центральная предельная теорема, m- зависимая случайная величина.

Theorem. Let $\{X_{n,i}\}$ be a triangular array of mean zero random variables. For each $n=1,2,\dots$ let $d = d_n$, $m = m_n$ and suppose $X_{n,1}, \dots, X_{n,d}$ is an m-dependent sequence of random variables. Define

$$B_{n,k,a}^2 = \text{Var} \left(\sum_{i=a}^{a+k-1} X_{n,i} \right),$$

$$B_n^2 \equiv B_{n,d,1} \equiv \text{Var} \left(\sum_{i=1}^d X_{n,i} \right).$$

Assume the following conditions hold. For some $\delta > 0$ and some $-1 \leq \gamma < 1$:

$$E|X_{n,i}|^{2+\delta} \leq \Delta_n \text{ for all } i, \tag{1}$$

$$B_{n,k,a}^2 / (k^{1+\gamma}) \leq K_n \text{ for all } a \text{ and for all } k \geq m, \tag{2}$$

$$B_n^2 / (dm^{\gamma+1}) \geq L_n, \tag{3}$$

$$K_n / L_n = O(1), \tag{4}$$

$$\Delta_n / L_n^{(2+\delta)/2} = O(1) \tag{5}$$

$$m^{1+(1-\gamma)(1+2/\delta)} / d \rightarrow 0 \tag{6}$$

Then, $B_n^{-1}(X_{n,1} + \dots + X_{n,d}) \Rightarrow N(0,1)$

Proof of Theorem . In the proof we will need a result for bounding moments of m-dependent sequences. We will state it as a corollary of the following lemma, which implicitly is given in Chow and Teicher (1978) and deals with independent sequences.

Lemma A.1. Let $\{Y_i\}$ be an independent sequence of mean zero random variables. Assume $E|Y_i|^q \leq \Delta$ for some $q \geq 2$ and all i .

$$\text{Then, } E \left| \sum_{i=1}^n Y_i \right|^q \leq C_q^q \Delta n^{q/2}$$

Where C_q is a positive constant depending only upon q .

Proof. See Theorem 2 and Corollary 2 in Section 10:3 of Chow and Teicher (1978).

Corollary A.1. Let $\{X_i\}$ be an m -dependent sequence of mean zero random variables. Assume $E|X_i|^q \leq \Delta$ for some $q \geq 2$ and all i .

Then, for all $n \geq 2m$,

$$E \left| \sum_{i=1}^n X_i \right|^q \leq C_q^q \Delta (4mn)^{q/2}$$

where C_q is a positive constant depending only upon q .

Proof. Define $t = [n/m]$ where $[\cdot]$ denotes the integer part. Now split $X_1 + \dots + X_n$ into t blocks of size m and a remainder block: $X_1 + \dots + X_n \equiv A_1 + \dots + A_t + A_{t+1}$. Due to m -dependence, the odd-numbered blocks are independent of each other, as are the even-numbered blocks. This allows us to apply Lemma A.1:

$$\begin{aligned} \left\| \sum_{i=1}^n X_i \right\|_q &\leq \left\| \sum_{i \text{ odd}} A_i \right\|_q + \left\| \sum_{i \text{ even}} A_i \right\|_q \quad (\text{by Minkowski}) \\ &\leq 2C_q m (\Delta)^{1/q} (t/2 + 1)^{1/2} \quad (\text{by Lemma A:1 and Minkowski}): \end{aligned}$$

But, this is equivalent to

$$\begin{aligned} E \left| \sum_{i=1}^n X_i \right|^q &\leq C_q^q 2^q m^q \Delta (t/2 + 1)^{q/2} \\ &\leq C_q^q 2^q m^q \Delta (t)^{q/2} \leq C_q^q 2^q \Delta (mn)^{q/2} = C_q^q \Delta (4mn)^{q/2}. \end{aligned}$$

We are now able to prove the theorem. The main idea of the proof follows Berk (1973), but we need some modifications, since our theorem is more general.

For each n , we choose an integer $p = p_n > 2m$ so that

$$\lim_{n \rightarrow \infty} m/p = 0, \quad \lim_{n \rightarrow \infty} p^{1+(1-\gamma)(1+2/\delta)} / d = 0. \tag{7}$$

This can be done, for example, by remembering assumption (6) and choosing p to be the smallest integer greater than $2m$ and greater than $m^{1/2} d^{1/2\xi}$, where ξ is equal to $1 + (1 - \gamma)(1 + 2 / \delta)$. Next, define integers $t = t_n$ and $q = q_n$ by $d = pt + q$, $0 \leq q < p$. The main idea of the proof is to split the sum $X_{n,1} + \dots + X_{n,d}$ into alternate blocks of length $p - m$ (the big blocks) and m (the little blocks). This is a common approach to proving central limit theorems for dependent random variables, and is attributed to Markov in Bernstein (1927). Let

$$U_{n,i} = X_{n,(i-1)p+1} + \dots + X_{n,ip-m}, \quad 1 \leq i \leq t,$$

$$V_{n,i} = X_{n,ip-m+1} + \dots + X_{n,ip}, \quad 1 \leq i \leq t$$

$$U_{n,t+i} = X_{n,tp+1} + \dots + X_{n,d}.$$

By definition, $X_{n,1} + \dots + X_{n,d} = \sum_{i=1}^{t+1} U_{n,i} + \sum_{i=1}^t V_{n,i}$. Since the $X_{n,i}$ are m -dependent and $p > 2m$, $\{U_{n,i}\}$ and $\{V_{n,i}\}$ are each independent sequences. It is easily seen that the difference between $B_n^{-1}(X_{n,1} + \dots + X_{n,d})$ and has variance approaching zero. Indeed,

$$\begin{aligned} \text{Var}\left(B_n^{-1} \sum_{i=1}^{t+1} V_{n,i}\right) &= B_n^{-2} \sum_{i=1}^l \text{Var}(V_{n,i}) \\ &\leq B_n^{-2} t \left[\sup_i \text{Var}(V_{n,i}) \right] \leq B_n^{-2} t K_n m^{1+\gamma} \quad (\text{by assumption (2)}) \\ &\leq B_n^{-2} (d/p) K_n m^{1+\gamma} \\ &\leq \frac{K_n}{L_n} \frac{m}{n} \rightarrow 0 \quad (\text{by assumption (3) and (4)}). \end{aligned}$$

Hence, provided they exist, the asymptotic distributions of the two quantities $B_n^{-1} \sum_{i=1}^{t+1} U_{n,i}$ and $B_n^{-1} \sum_{i=1}^d X_{n,i}$ are the same, and the goal now is to show that

$$B_n^{-1} \sum_{i=1}^{t+1} U_{n,i} \Rightarrow N(0,1).$$

In order to apply assumption (3) again, we will first establish that $B_n^{-2} \text{Var}\left(\sum_{i=1}^{t+1} U_{n,i}\right)$

tends to one, or, equivalently, $B_n^{-2} \text{Cov}\left(\sum_{i=1}^{t+1} U_{n,i}, \sum_{i=1}^t V_{n,i}\right)$ tends to zero. Note first that $\text{Cov}(U_{n,i}, V_{n,i}) = 0$ unless $j = i$ or $i - 1$. Furthermore,

$$\begin{aligned} \left| \text{Cov}(U_{n,i}, V_{n,i}) \right| &= \left| E(U_{n,i}, V_{n,i}) \right| \leq \left[\text{Var}(U_{n,i}) \text{Var}(V_{n,i}) \right]^{1/2} \\ &\leq K_n (mp)^{(1+\gamma)/2} \quad (\text{by assumption (2)}). \end{aligned}$$

Combining these two facts, we obtain

$$\left| \text{Cov}\left(\sum_{i=1}^{t+1} U_{n,i}, \sum_{i=1}^t V_{n,i}\right) \right| \leq 2K_n (mp)^{(1+\gamma)/2}$$

and finally,

$$\begin{aligned} B_n^{-2} \text{Cov}\left(\sum_{i=1}^{t+1} U_{n,i}, \sum_{i=1}^t V_{n,i}\right) &\leq 2 \frac{K_n}{L_n} \frac{t}{dm^\gamma} (mp)^{(1+\gamma)/2} \\ &\leq 2 \frac{K_n}{L_n} \frac{1}{pm^\gamma} (mp)^{(1+\gamma)/2} \\ &= 2 \frac{K_n}{L_n} \left(\frac{m}{p}\right)^{(1-\gamma)/2} \rightarrow 0 \quad (\text{by assumption (4) and since } \gamma < 1). \end{aligned}$$

By Lyapounov's theorem, it will now suffice to verify that $\sum_{i=1}^{t+1} E|U_{n,i}|^{2+\delta} / B_n^{2+\delta}$ tends to zero. By Corollary A.1,

$$E|U_{n,i}|^{2+\delta} \leq C_{2+\delta} \Delta_n (4pm)^{(2+\delta)/2}, \quad 1 \leq i \leq t+1,$$

And therefore

$$\sum_{i=1}^{t+1} E|U_{n,i}|^{2+\delta} / B_n^{2+\delta} \leq Const. \Delta_n (d/p+1)(pm)^{(2+\delta)/2} / B_n^{2+\delta}$$

By assumption (3), finally,

$$\Delta_n (d/p)(pm)^{(2+\delta)/2} / B_n^{2+\delta} \leq \Delta_n L_n^{-(2+\delta)/2} \frac{d}{p} \left(\frac{pm}{dm^\gamma} \right)^{(2+\delta)/2}$$

$$\leq \Delta_n L_n^{-(2+\delta)/2} \left(\frac{p}{d} \right)^{\delta/2} m^{(1-\gamma)(2+\delta)/2}$$

$$= O(1)AB \quad (\text{by assumption (5)});$$

where $A = p^{\delta/2+(1-\gamma)(2+\delta)/2} d^{-\delta/2}$ and $B = \left(\frac{m}{p} \right)^{(1-\gamma)(2+\delta)/2}$. The second condition on

p in (7) implies that A tends to zero. The first condition on p in (7), together with the fact that $\gamma \leq 1$, imply that B tends to zero as well.

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