THE ORTHOGONAL GROUP OF THE TWO-DIMENSIONAL BILINEAR-METRIC SPACE WITH THE FORM $x_1y_1 + 2x_2y_2$ OVER THE FIELD OF RATIONAL NUMBERS

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Abstract. Let Y be the 2-dimensional bilinear-metric space, O(Y) be the group of all orthogonal transformations of Y Put MO(Y) $= \{F$: Y → Y $|Fx = gx + b, g \in O$(Y)$, b \in$ Y$\}$, SO(Y) $= \{g \in O$(Y)$|detg = 1\}$ and MSO(Y) $= \{F \in MO$(Y)$|detg = 1\}$.The present paper is devoted to solutions of problems of G-equivalence of m-tuples in Y for groups $G = O$(Y)$, SO$(Y)$, MO$(Y), MSO(Y). Complete systems of G-invariants of m-tuples in Y for these groups are obtained. Complete systems of relations between elements of the obtained complete systems of G-invariants are given for these groups

Абстракт. Пусть Y— двумерное билинейно-метрический пространство, O(Y) — группа всех ортогональных преобразований Y Положим MO(Y)={F:Y→Y|Fx=gx+b,g∈O(Y),b∈Y}, SO(Y)={g∈O(Y)|detg=1} и MSO(Y) $= \{F \in MO$(Y)$|detg = 1\}$M. Настоящая работа посвящена решению проблем G-эквивалентности m-кортежей в Yдля групп G=O(Y), SO(Y), MO(Y), MSO(Y). Получены полные системы G-инвариантов m-кортежей в Yдля этих групп. Для этих групп даны полные системы отношений между элементами полученных полных систем G-инвариантов.

Abstract. Y ikki o‘lchovli ikki chiziqli metrik fazo, O(Y) -Y barcha ortogonal akslanishlar gruppasi bo‘lsin. MO(Y)={F:Y→Y|Fx=gx+b,g∈ O(Y) ,b∈Y}, SO(Y)={g∈O(Y)|detg=1} va MSO(Y) $= \{F \in MO$(Y)$|detg = 1\}$.Bu ish G=O(Y), SO(Y), MO(Y), MSO(Y) gruppa uchun Y dagi m-ketma-ketliklarning G-ekvivalentligi masalalarini yechishga bag‘ishlangan. Bu gruppalar uchun Y dagi m-ketma-ketliklarning G-invariantlarining to'la sistemalari olinadi. Bu gruppalar uchun G-invariantlarning olingan to‘la sistemalari elementlari orasidagi munosabatlarning to‘la sistemalari berilgan.

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Introduction. In life, in science, in technology, and especially in computer science, many problems are reduced to problems on rational numbers. In particular, this applies to geometric problems over the field of rational numbers. Therefore, there is a great need today to develop geometries over rational numbers. The geometries on real numbers and complex numbers are well developed. The geometries on rational numbers are very little developed. Even two-dimensional geometries on rational numbers are poorly developed. In this paper, two-dimensional geometries on rational numbers are developed using the methods of invariance theory. The idea of developing geometry using the methods of the invariant theory was first proposed in 1872 in the Erlangen program of the German mathematician Felix Klein. In this article, we have developed two-dimensional geometry on rational numbers. The results obtained from the development of geometries over the field of rational numbers are expected to be useful in computational geometry, computer algebra, computer geometry, computer graphics, and computer-assisted image recognition. In the book [[1], Proposition 9.7.1], for the group of motions in the Euclidean geometry, the orbit of m vectors is characterized by distances between m-vectors. A complete system of relations between elements of this complete system is also given in [[1], Theorem 9.7.3.4]. In the paper [3], a complete system of invariants of m-tuples in the two-dimensional pseudo-Euclidean geometry of index 1 and a complete system relations between the obtained complete system of invariants are given. In the paper [4], a complete system of invariants of m-tuples in the one-dimensional projective space and a complete system relations between the obtained complete system of invariants are given. General theory of m-point invariants considered in the invariant theory (see [2]).

A linear representation of the field of $Q(\sqrt{-2})$ in two-dimensional rational space

Let $Y = Q(\sqrt{-2})$ be the two-dimensional metric space over the field Q of rational numbers with the bilinear form $x_1y_1 + 2x_2y_2$. For $x = x_1 + x_2\sqrt{-2}, y = y_1 + y_2\sqrt{-2} \in Y$, we put $\langle x, y\rangle = x_1y_1 + 2x_2y_2$. Then $\langle x, y\rangle$ is a bilinear form on Y and $\langle x, x\rangle = x_1^2 + 2x_2^2$ is a quadratic form on Y. For convenience, we denote by $\varphi(x)$ the quadratic form $\langle x, x\rangle$.

Let $x = x_1 + x_2\sqrt{-2} \in Y$. Denote by M_x the matrix of the form

$$M_x = \begin{pmatrix} x_1 & -2x_2 \\ x_2 & x_1 \end{pmatrix}. \quad (0.1)$$

Denote by M_Y the set of all matrices M_x, where $x \in Y$. We consider on the set M_Y standard matrix operations: the addition and the multiplication of matrices. Then it is easy to see that M_Y is a field with the unit element, where the unit element is the unit matrix.

Proposition 1. *$\{M_Y, +, \cdot\}$ is a field.*

Proof. It is easy and it is omitted.

Proposition 2. *The mapping $M: Y \to M_Y$, where $M: x \to M_x, \forall x \in Y$, is an isomorphism of fields Y and M_Y.*

Proof. It is easy and it is omitted.

We write elements $x = x_1 + \sqrt{-2}x_2 \in Y, y = y_1 + \sqrt{-2}y_2 \in Y$ in column forms: $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then we write the product of these elements in Y in the column form as follows:

$$x \cdot y = \begin{pmatrix} x_1y_1 - 2x_2y_2 \\ x_1y_2 + x_1y_2 \end{pmatrix}. \quad (0.2)$$

The product of M_x and the column y has the following form

$$M_x y = \begin{pmatrix} x_1 & -2x_2 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 - 2x_2 y_2 \\ x_1 y_2 + x_1 y_2 \end{pmatrix}. \quad (0.3)$$

From the equations (0.2) and (0.3) we obtain the following equation:

$$x \cdot y = \begin{pmatrix} x_1 y_1 - 2x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} = \begin{pmatrix} x_1 & -2x_2 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = M_x y. \quad (0.4)$$

Let $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 1. *The element $W(x) = x_1 - \sqrt{-2}x_2$ is called a conjugate element of $x = x_1 + \sqrt{-2}x_2$.*

Proposition 3. *The following equalities $\varphi(x) = det(M_x)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ hold for all $x, y \in Y$.*

Proof. It is easy and it is omitted.

Proposition 4. *The mapping $W: x \to x$ is an involution of the field Y and the following equalities $x + W(x) = 2x_1, \langle x, x \rangle = xW(x) = W(x)x = x_1^2 + 2x_2^2 = \varphi(x)$ hold for all $x \in Y$.*

Proof. It is easy and it is omitted.

Proposition 5. *The function $\varphi(x)$ has the following properties:*

1. $\varphi(\lambda x) = \lambda^2 \varphi(x), \forall \lambda \in Q, \forall x \in Y$;
2. $\varphi(e) = 1$ *for the unit element* $e \in Y$;
3. $\varphi(x) = \varphi(W(x)) = xW(x) = W(x)x, \forall x \in Y$;
4. $\langle x, y \rangle = \langle W(x), W(y) \rangle, \forall x, y \in Y$.

Proof. It is easy and it is omitted.

Proposition 6. *Let $x \in Y$. Then the element $x^{-1} \in Y$ exists if and only if $\varphi(x) \neq 0$. In the case $\varphi(x) \neq 0$, the equalities $x^{-1} = \frac{W(x)}{\varphi(x)}$ and $\varphi(x^{-1}) = \frac{1}{\varphi(x)}$ hold.*

Proof. It is easy and it is omitted.

Put $G(Y) = \{x \in Y: \varphi(x) \neq 0\}$ and $SG(Y) = \{x \in Y: \varphi(x) = 1\}$. $G(Y)$ is a group with respect to the multiplication in Y. $SG(Y)$ is a subgroup of the group $G(Y)$. Put $GM_Y = \{M_a | a \in G(Y)\}$ and $SGM_Y = \{M_a | a \in SG(Y)\}$. SGM_Y is a subgroup of the group GM_Y and the mapping $M: SG(Y) \to SGM_Y$, where $M(a) = M_a$, is a linear representation of the group $SG(Y)$ in Q^2.

A description of all orthogonal transformations of the space $Y = Q(\sqrt{-2})$

Definition 2. *A transformation $F: Q^2 \to Q^2$ is called orthogonal if $\langle F(x), F(y) \rangle = \langle x, y \rangle, \forall x, y \in Q^2$.*

We denote the set of all orthogonal transformation of Q^2 by $O(Y)$.

Let $F, T \in O(Y)$. Put $(F \circ T)(x) = F(T(x)), \forall x \in Q^2$. It is easy to see that $F \circ T \in O(Y)$ that is the composition of the orthogonal transformations is also orthogonal.

Theorem 1. *Every orthogonal transformation is linear.*

Proof. Let $T \in O(Y)$. Consider vectors $e_1 = (1;0)$ and $e_2 = (0;1)$. Then following equalities hold: $\langle e_1;e_1\rangle = 1$, $\langle e_1;e_2\rangle = 0$, $\langle e_2;e_2\rangle = 2$, $\langle T(e_1),T(e_2)\rangle = \langle e_1,e_2\rangle = 0$, $\langle T(e_1),T(e_1)\rangle = \langle e_1,e_1\rangle = 1$ and $\langle T(e_2),T(e_2)\rangle = \langle e_2,e_2\rangle = 2$. These equalities imply that vectors $T(e_1)$ and $T(e_2)$ is an orthogonal basis in Y. Let $x = x_1e_1 + x_2e_2 \in Q^2$ be an arbitrary element. Then $T(x) = c_1T(e_1) + c_2T(e_2)$ for some $c_1 \in Q, c_2 \in Q$. In this case, we have $\langle x,e_1\rangle = \langle x_1e_1 + x_2e_2, e_1\rangle = \langle x_1e_1,e_1\rangle + \langle x_2e_2,e_2\rangle = x_1\langle e_1,e_1\rangle + x_2\langle e_2,e_1\rangle = x_1$. Similarly we obtain $\langle x,e_2\rangle = x_2$. Since $T(e_1)$ and $T(e_2)$ is an orthogonal basis in Y, we obtain $\langle T(x),T(e_1)\rangle = \langle c_1T(e_1) + c_2T(e_2),T(e_1)\rangle = \langle c_1T(e_1),T(e_1)\rangle + \langle c_2T(e_2),T(e_2)\rangle = c_1\langle T(e_1),T(e_1)\rangle + c_2\langle T(e_2),T(e_1)\rangle = c_1$. Using the above equalities and the orthogonality of the transformation T, we obtain $c_1 = \langle T(x),T(e_1)\rangle = \langle x,e_1\rangle = x_1$. Similarly we obtain $c_2 = x_2$. These equalities imply the following equality $T(x) = x_1T(e_1) +_2 T(e_2)$ for an arbitrary $x = x_1e_1 + x_2e_2 \in Q^2$. Using this equality, we check the linearity conditions:

1. $T(x+y) = (x_1+y_1)T(e_1) + (x_2+y_2)T(e_2) = x_1T(e_1) + y_1T(e_1) + x_2T(e_2) + y_2T(e_2) = \ = (x_1T(e_1) + x_2T(e_2)) + (y_1T(e_1) + y_2T(e_2)) = T(x) + T(y)$. So $T(x+y) = T(x) + T(y), \forall x,y \in Q^2$ fits.

2. $\lambda x = \lambda x_1e_1 + \lambda x_2e_2$ This equality imply the following equality $T(\lambda x) = \lambda x_1T(e_1) + \lambda x_2T(e_2) = \lambda(x_1T(e_1) + x_2T(e_2)) = \lambda T(x), \forall x \in Y, \forall \lambda \in Q$,. Hence $T(\lambda x) = \lambda T(x), \forall x \in Q^2, \forall \lambda \in Q$.

The theorem 1 is proved.

Proposition 8. *Let* $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ *be column vectors in* Q^2. *Then*

$$\langle x,y\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1y_1 + 2x_2y_2 \qquad (0.5)$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T = (x_1 \ \ x_2)$.

Proof. Let $x,y \in Q^2$. Then $\langle x,y\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1 \quad x_2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1 \quad -2x_2) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1y_1 + 2x_2y_2$.

Since orthogonal transformation is linear, we can express the transformation by a matrix on a basis.

Proposition 9. *Let* $C \in O(Y)$. *Then* $det(C) = 1$ *or* $det(C) = -1$.

We denote by $SO(Y)$ the set of all matrices $C = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \in O(Y)$ such that $a,b \in Q$ and $det(C) = 1$.

Proposition 10. 1) *The set* $O(Y)$ *is a group with respect to the matrices multiplication and* $SO(Y)$ *is a subgroup of* $O(Y)$.

2) *The equality* $O(Y) = SO(Y) \cup \{CW|C \in SO(Y)$ *holds, where* CW *is the multipcation of matrices* C *and* W.

Proof. It follows from Propositions 5 and 9.

We put $GM_Y = \left\{ \begin{pmatrix} a & -2b \\ b & a \end{pmatrix} \mid \begin{vmatrix} a & -2b \\ b & a \end{vmatrix} \neq 0, a, b \in Q \right\}$ and $SM_Y = \{C \in GM_Y \mid \det(C) = 1\}$.

There is a question. Is it possible to find evident forms of all orthogonal transformations? We can answer this question completely by the following theorem.

Theorem 2. *The equality* $SO(Y) = SM_Y$ *holds and* $SO(Y) = \left\{ \begin{pmatrix} a & -2b \\ b & a \end{pmatrix} \mid a^2 + 2b^2 = 1, a, b \in Q \right\}$.

Proof. Assume that $C = \|c_{is}\|_{i,s=1,2} \in SM_Y$, where $c_{11} = c_{22} = a$, $c_{12} = -2b, c_{21} = b$. We will prove that $C = \|c_{is}\|_{i,s=1,2} \in SO(Y)$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in Q^2$. We have $C(x) = \begin{pmatrix} ax_1 - 2bx_2 \\ bx_1 + ax_2 \end{pmatrix}$, $C(y) = \begin{pmatrix} ay_1 - 2by_2 \\ by_1 + ay_2 \end{pmatrix}$. Using $a^2 - 2b^2 = 1$, we obtain $\langle C(x), C(y) \rangle = (ax_1 - 2bx_2)(ay_1 - 2by_2) + 2(bx_1 + ax_2)(by_1 + ay_2) = a^2x_1y_1 + 2ab(x_2y_1 + x_1y_2) + 2^2b^2x_2y_2 - 2b^2x_1y_1 - 2ab(x_2y_1 + x_1y_2) - 2a^2x_2y_2 = x_1y_1(a^2 + 2b^2) + 2x_2y_2(a^2 + 2b^2) = x_1y_1 + 2x_2y_2 = \langle x, y \rangle$. Hence $C \in SO(Y)$.

Conversely, assume that $C \in SO(Y)$, where $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, $\det(C) = c_{11}c_{22} - c_{12}c_{21} = 1$. Then $C(x) = \begin{pmatrix} c_{11}x_1 + c_{12}x_2 \\ c_{21}x_1 + c_{22}x_2 \end{pmatrix}$, $C(y) = \begin{pmatrix} c_{11}y_1 + c_{12}y_2 \\ c_{21}y_1 + c_{22}y_2 \end{pmatrix}$ $\langle C(x), C(y) \rangle = \langle x, y \rangle$ for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in Q^2$. Using equalities $\det(C) = c_{11}c_{22} - c_{12}c_{21} = 1$, $\langle C(x), C(y) \rangle = \langle x, y \rangle$, $\langle x, y \rangle = x_1y_1 - 2x_2y_2$ and $\langle C(x), C(y) \rangle = (c_{11}^2 - 2c_{21}^2)x_1y_1 + (c_{12}^2 + 2c_{22}^2)x_2y_2 + (c_{11}c_{12} + 2c_{22}c_{21})x_1y_2 + (c_{11}c_{12} - 2c_{22}c_{21})x_2y_1$ for all $x, y \in Q^2$, we obtain the following equalities:

(I). $c_{11}c_{22} - c_{12}c_{21} = 1$;

(II). $c_{11}^2 - 2c_{21}^2 = 1$;

(III). $c_{12}^2 - 2c_{22}^2 = -2$;

(IV). $c_{11}c_{12} - 2c_{22}c_{21} = 0$.

We consider the following two cases: (i) $c_{12} = 0$ and (ii) $c_{12} \neq 0$.

Let (i) $c_{12} = 0$. Then the equality (III) implies $-2c_{22}^2 = -2$. Hence $c_{22} = 1$ or $c_{22} = -1$.

Let $(i.1)$ $c_{22} = 1$. Then the equalities $(IV), (I)$ imply $c_{21} = 0$ and $c_{11} = 1$.

Let $(i.2)$ $c_{22} = -1$. Then the equalities $(IV), (I)$ imply $c_{21} = 0$ and $c_{11} = -1$. Hence, in this case, we obtain only the following two matrices: $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SM_Y$ and $C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SM_Y$.

(ii) Let $c_{12} \neq 0$. Then the equality (IV) implies $c_{11} = \frac{2c_{22}c_{21}}{c_{12}}$. In this case, using (II), we obtain the following implications: $\left(\frac{2c_{22}c_{21}}{c_{12}}\right)^2 - 2c_{21}^2 = 1 \Rightarrow$

$2^2c_{22}^2c_{21}^2 - 2c_{12}^2c_{21}^2 = c_{12}^2 \Rightarrow 2c_{21}^2(2c_{22}^2 - c_{12}^2) = c_{12}^2 \overset{(3)}{\Rightarrow} (2c_{21})^2 = c_{12}^2$. Hence $(ii.1)$ $c_{12} = 2c_{21}$ or $(ii.2)$ $c_{12} = -2c_{21}$.

$(ii.1)$ Let $c_{12} = 2c_{21}$. Then the equality (IV) implies $c_{11}c_{12} - 2c_{22}c_{21} = 0 \Rightarrow 2c_{21}(c_{11} - c_{22}) = 0 \Rightarrow (c_{12} \neq 0 \Rightarrow c_{21} \neq 0) \Rightarrow c_{11} - c_{22} = 0 \Rightarrow c_{11} = c_{22}$. Hence $C_3 = \begin{pmatrix} c_{22} & 2c_{21} \\ c_{21} & c_{22} \end{pmatrix} \in SM_Y$.

$(ii.2)$ Let $c_{12} = -2c_{21}$. Using equality (IV), we obtain the following implications: $2c_{21}(c_{11} + c_{22}) = 0 \Rightarrow (c_{12} \neq 0 \Rightarrow c_{21} \neq 0) \Rightarrow c_{11} + c_{22} = 0 \Rightarrow c_{11} = -c_{22}$. Hence we have $c_{12} = -2c_{21}$ and $c_{11} = -c_{22}$. These equalities and the equality (I) imply the following equality $-c_{11}^2 - (-2c_{21})c_{21} = 1$. This equality implies $c_{11}^2 - 2c_{21}^2 = -1$. But, last equality is a contradiction to the equality (II). Hence $c_{12} \neq -2c_{21}$. This means that inthe case (ii.2), the matrix C is not element of $SO(Y)$. The theorem is completed.

Hence, we conclude from the above theorem that every special orthogonal transformations will be matrices $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$, such that $a^2 - 2b^2 = 1, a, b \in Q$. In that case, is the solution of the equation $a^2 - 2b^2 = 1$ in the rational numbers field? We can answer this question by the following theorem.

Theorem 3. *The description of the elements of the group $SO(Y)$ is as follows.*

(i). *There is no element $x = (x_1, x_2) \in Q^2$, such that $x_1 = 0$ and $M_x \in SO(Y)$. There are only two elements $(x_1, x_2) \in Q^2$, such that $x_2 = 0$ and $M_x \in SO(2, Q)$. These are $(1,0)$ and $(-1,0)$.*

(ii). *Assume that $x = (x_1, x_2) \in Q^2$ such that $x_2 \neq 0$ and $M_x \in SO(Y)$. Then there is the number $r \in Q$, where $r \neq 0$, such that the equalities are satisfied:*

$$x_1 = \frac{2+r^2}{2-r^2}, \qquad x_2 = \frac{2r}{2-r^2}. \tag{0.6}$$

(iii). *Conversely, assume that r is an arbitrary nonzero element in Q and for $x = (x_1, x_2) \in Q^2$ the equalities are satisfied (0.6). Then $M_x \in SO(Y)$.*

Proof. (i). This is obvious.

(ii). Assume that $x = (x_1, x_2) \in Q$ such that $x_2 \neq 0$ and $x_1^2 - 2x_2^2 = 1$.

First, we prove that in this case $x_1^2 \neq 1$. Suppose $x_1^2 = 1$. Then from the equation $x_1^2 - 2x_2^2 = 1$, we obtain that $x_2^2 = 0$. It follows that $x_2 = 0$. This contradicts to $x_2 \neq 0$. So we proved $x_1^2 \neq 1$, $x_1 \neq 1$ and $x_1 \neq -1$.

From the equation $x_1^2 - 2x_2^2 = 1$ and from the inequalities $x_1 \neq 1$, $x_1 \neq -1$ we obtain the following equalities: $x_1^2 - 1 = 2x_2^2 \Rightarrow 2x_2^2 = (x_1 + 1)(x_1 - 1) \Rightarrow \frac{2x_2}{1+x_1} = \frac{x_1-1}{x_2}$.

Put $r = \frac{2x_2}{1+x_1}$. Then we have $r = \frac{x_1-1}{x_2}$. From these two equalities we obtain the following equalities $\frac{1}{x_2} + \frac{x_1}{x_2} = \frac{2}{r}, \frac{x_1}{x_2} - \frac{1}{x_2} = r$. From last equalities we obtain

$\frac{2}{x_2}=\frac{2}{r}-r, \frac{2x_1}{x_2}=\frac{2}{r}+r$. We find x_1, x_2 from these two equalities and we obtain the following equalities $x_1=\frac{2+r^2}{2-r^2}, x_2=\frac{2r}{2-r^2}$. The (ii) is proved.

(iii). Conversely, let $r \in Q$ be an arbitrary rational number other than zero. Put $x_1=\frac{2+r^2}{2-r^2}$, $x_2=\frac{2r}{2-r^2}$. We have $x_1^2-2x_2^2=(\frac{2+r^2}{2-r^2})^2-2(\frac{2r}{2-r^2})^2=\frac{2^2+4r^2+r^4-8r^2}{(2-r^2)^2}=\frac{4-4r^2+r^4}{(2-r^2)^2}=1$. Therefore, $M_x \in SO(Y)$.

Hence, all special orthogonal matrices are as follows:

$$SO(Y)=\left\{\begin{pmatrix}\frac{2+r^2}{2-r^2} & \frac{4r}{2-r^2}\\ \frac{2r}{2-r^2} & \frac{2+r^2}{2-r^2}\end{pmatrix} \mid \ \forall r \in Q, r \neq 0\right\} \qquad (0.7)$$

and all orthogonal matrices are as follows:

$$O(Y)=\left\{\begin{pmatrix}\frac{2+r^2}{2-r^2} & \frac{4r}{2-r^2}\\ \frac{2r}{2-r^2} & \frac{2+r^2}{2-r^2}\end{pmatrix} \mid \ \forall r \in Q, r \neq 0\right\} \cup \left\{\begin{pmatrix}\frac{2+r^2}{2-r^2} & \frac{4r}{2-r^2}\\ -\frac{2r}{2-r^2} & -\frac{2+r^2}{2-r^2}\end{pmatrix} \mid \ \forall r \in Q, r \neq 0\right\}. \qquad (0.8)$$

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