A SUFFICIENT CONDITION FOR THE POSSIBILITY OF COMPLETING THE PURSUIT

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Annotation. In this article, we have considered a simple motion differential game of pursuers and one evader in. Here controls of the pursuers are subjected to linear constraints which is the generalization of both integral and geometrical constraints, and control of the evader is subjected to a geometrical constraint. To solve a pursuit problem, the attainability domain of each pursuer has been constructed and therefore, necessary and sufficient conditions have been obtained by intersection of them.

Key words: differential equation, differential game, integral boundary, geometric boundary, control, control function.

Аннотация. В этой статье мы рассмотрели простую игру с дифференциальным движением преследователей и одного убегающего. Здесь управление преследователями подчиняется линейным ограничениям, что является обобщением как интегральных, так и геометрических ограничений, а управление убегающим подчиняется геометрической ограничение. Для решения задачи преследования построена область достижимости каждого преследователя и, следовательно, получены необходимые и достаточные условия их пересечения.

Ключевые слова: дифференциальное уравнение, дифференциальная игра, интегральная граница, геометрическая граница, управление, функция управления.

Annotatsiya. Ushbu maqolada biz ta'qibchilarning differentsial harakati va bitta qochish bilan oddiy o'yinni ko'rib chiqdik. Bu erda ta'qibchilarning nazorati chiziqli cheklovlarga bo'ysunadi, bu ham integral, ham geometrik cheklovlarning umumlashtirilishi va qochishni boshqarish geometrik cheklovga bog'liq. Ta'qib qilish muammosini hal qilish uchun har bir ta'qibchining erishish mumkin bo'lgan hududi quriladi va shuning uchun ularning kesishishi uchun zarur va etarli shartlar olinadi.

Kalit so'zlar: differensial tenglama, differensial o'yin, integral chegara, geometrik chegara, nazorat, boshqaruv funksiyasi.

Differential games are a mathematical object that models conflict situations described by differential equations.

Let us give a more detailed description of the formalization of differential games proposed by L.S. Pontryagin [1]. Let the motion of a conflict-controlled system in the phase vector $z \in R^n, n \geq 1$, be described by the differential equation

$$z = F(z, u, v),$$

where u, v are the control parameters (control vectors) of the first (pursuing, catching up) and second (pursuing, escaping) players, respectively, $u \in U, v \in V, U$ and V are non-empty subsets of the Euclidean spaces R^p and R^q respectively, $F: R^n \times U \times V \to R^n$ is a continuous function. In addition, a non-empty target set M (terminal set) is specified in R^n. L.S. Pontryagin emphasizes: "We associate two different problems with the differential game.

1. Our goal is to complete the game, i.e. bringing the point z to the set M; at the same time, to achieve this goal, we have at our disposal the controlling parameter of the pursuing u, so that at each time $t \geq 0$ we choose the value u(t) of this parameter using the functions $z(s)$ and v(s) on the segment $t - \theta \leq s \leq t$, where θ is a suitably chosen positive number. These are the rules of the pursuit game.

2. Our goal is to prevent the end of the game, i.e. preventing the arrival of the point z on the set M; at the same time, to achieve this goal, we have at our disposal the control parameter of the pursued v, so that at each time $t \geq 0$ we choose the value $v(t)$ of this parameter using the functions $z(s)$ and $u(s)$ on the segment $t - \theta \leq s \leq t$. These are the rules of the game of escape" [1].

We consider the differential game of pursuit

$$\dot{z} = Cz - u + v, (1)$$

where $z \in R^n, C$ is a linear mapping of the space R^n into itself, $|u| \leq 1, \|v(\cdot)\| = \sigma, \sigma$ is a positive constant. The terminal set has the form

$$M = \{z: z \in R^n, |\pi z| \leq l\},$$

Where π – orthogonal projection operator R^n on L, L – some subspace R^n, l – given positive number.

Definitions. We will say that in game (1) from the starting point (initial position) $z_0 \in R^n \backslash M$ you can complete the pursuit in time $T(z_0)$, if there is a function $u = u(t, v), 0 \leq t \leq T(z_0), v \in V, u(t, v) \in U$, such that for an arbitrary measurable function $v = v(t), 0 \leq t \leq T(z_0), v(t) \in V$, function $u(t, v(t)), 0 \leq t \leq T(z_0)$, is measurable and the trajectory$z = z(t), 0 \leq t \leq T(z_0)$, equations $z = Cz - u(t, v(t)) + v(t), z(0) = z_0$, falls on M, т.е. $z(t) \in M$ at some $t = t' \in [0, T(z_0)]$. Number $T(z_0)$ is called the pursuit time from the point z_0, function $u(t, v), 0 \leq t \leq T(z_0), v \in V$, – persecution function.

The solution of the pursuit problem is understood as:

1) finding starting points $z_0 \in R^n \backslash M$, of which the completion of the pursuit in the sense of the definition introduced above is possible;

2) finding explicitly or specifying an algorithm for constructing the pursuit function $u(t, v)$.

Theorem. Let there be a positive constant d such that

$$\|e^{tC}\| \leq d \ (2)$$

for all $t > 0$. Then in the game (2.1) it is possible to complete the pursuit and an arbitrary point $z_0 \in R^n \backslash M$ during

$$T(z_0) = d|z_0| + d^4 K\theta^2(N-1),$$

where $K = \max(|z_0|, 1)$, $\theta = \max(\sigma, 1)$, $N = \left[\frac{cd^3(|z_0|+\sigma^2)}{l}\right]$, $c = \|\pi\|$, $[a]$ – the integer part of number a.

Proof. Let, $v(t), t \geq 0$, – arbitrary measurable function, $\|v(\cdot)\|, = \sigma$. Ha segment $[0, t_1]$ we will put

$$u(t) = \frac{1}{d|z_0|} e^{tC} z_0,$$

where $t_1 = d|z_0|$. This is a continuous function, hence measurable. Let us show that it satisfies $|u(t)| \leq 1$ for all $t \in [0, t_1]$:

$$|u(t)| = \frac{1}{d|z_0|} \cdot |e^{tC} z_0| \leq \frac{1}{d|z_0|} \cdot \|e^{tC}\| \cdot |z_0| \leq \frac{1}{d|z_0|} \cdot d|z_0| = 1.$$

Solution of equation (1) on the segment $[0, t_1]$ has the form

$$z(t) = e^{tC}\left[z_0 - \int_0^t e^{-rC} u(r) dr + \int_0^t e^{-rC} v(r) dr\right] =$$

$$= e^{tC}\left[z_0 - \int_0^t e^{-rC} \cdot \frac{1}{d|z_0|} e^{rC} z_0 dr + \int_0^t e^{-rC} v(r) dr\right] =$$

$$= e^{tC}\left[z_0 - \frac{t}{d|z_0|} \cdot z_0 + \int_0^t e^{-rC} v(r) dr\right].$$

Considering $t_1 = d|z_0|$, we have

$$z(t_1) = e^{t_1 C}\left[z_0 - \frac{t_1}{d|z_0|} \cdot z_0 + \int_0^{t_1} e^{-rC} v(r) dr\right] = \int_0^{t_1} e^{(t_1 - r)C} v(r) dr.$$

That's why

$$\pi z_1 = \pi z(t_1) = \int_0^{t_1} \pi e^{(t_1-r)C} v(r)dr\,.$$

Let's estimate $|z_1|$

$$|z_1| = \left|\int_0^{t_1} e^{(t_1-r)C} v(r)dr\right| < \int_0^{t_1} \left\|e^{(t_1-r)C}\right\| \cdot |v(r)|dr \le d \cdot \int_0^{t_1} |v(r)|dr \le$$

$$\le d\sqrt{t_1}\sigma_1 = d\sqrt{d|z_0|}\sigma_1 = d^{\frac{3}{2}}|z_0|^{\frac{1}{2}}\sigma_1 \le d^3 K\theta^2 \ (3)$$

Let's pretend that $z_1 \notin M$. Then we get

$$l < |\pi z_1| = \left|\int_0^{t_1} \pi e^{(t_1-r)C} v(r)dr\right| \le \|\pi\| \cdot \int_0^{t_1} \left\|e^{(t_1-r)C}\right\| \cdot |v(r)|dr.$$

Using the Cauchy-Bunyakovsky inequality [3], we have

$$l < cd\int_0^{t_1} |v(r)|dr \le cd\sqrt{\int_0^{t_1} dr} \cdot \sqrt{\int_0^{t_1} v^2(r)dr} =$$

$$= cd\sigma_1\sqrt{t_1} = cd\sigma_1\sqrt{d|z_0|} \le cd^3|z_0|^{\frac{1}{2}}\sigma_1 \ (4.1)$$

where $\sigma_1 = \sqrt{\int_0^{t_1} v^2(r)dr}$.

On the segment $[t_1, t_2]$ control the pursuer as follows:

$$u(t) = \frac{1}{d|z_1|} e^{(t-t_1)C} z_1,$$

where $t_2 = t_1 + d|z_1|$. It is easy to show that $|u(t)| \le 1$ при $t_1 \le t \le t_2$. Solution of equation (1) on the segment $[t_1, t_2]$ has the form

$$z(t) = e^{tC}\left[e^{-t_1 C} z_1 - \int_{t_1}^{t} e^{-rC} u(r)dr + \int_{t_1}^{t} e^{-rC} v(r)dr\right] =$$

$$= e^{tC}\left[e^{-t_1C}z_1 - \int_{t_1}^{t} e^{-rC}\cdot\frac{1}{d|z_1|}\cdot e^{(r-t_1)C}z_1dr + \int_{t_1}^{t} e^{-rC}v(r)dr\right] =$$

$$= e^{tC}\left[e^{-t_1C}z_1 - \frac{t-t_1}{d|z_1|}\cdot e^{-t_1C}z_1 + \int_{t_1}^{t} e^{-rC}v(r)dr\right].$$

Considering $t_2 = t_1 + d|z_1|$, we have

$$z(t_2) = e^{t_2C}\left[e^{-t_1C}z_1 - \frac{t_2-t_1}{d|z_1|}\cdot e^{-t_1C}z_1 + \int_{t_1}^{t_2} e^{-rC}v(r)dr\right] =$$

$$= \int_{t_1}^{t_2} e^{(t_2-r)C}v(r)dr.$$

That's why

$$\pi z_2 = \pi z(t_2) = \int_{t_1}^{t_2} \pi e^{(t_2-r)C}v(r)dr\,.$$

Suppose $z_2 \notin M$. Then we get

$$l < |\pi z_2| = \left|\int_{t_1}^{t_2} \pi e^{(t_2-r)C}v(r)dr\right| \le cd\sqrt{t_2-t_1}\cdot\sqrt{\int_{t_1}^{t_2} v^2(r)dr} =$$

$$= cd\sigma_2\sqrt{d|z_1|} \le cd^{\frac{3}{2}}|z_1|^{\frac{1}{2}}\sigma_2$$

where $\sigma_2 = \sqrt{\int_{t_1}^{t_2} v^2(r)dr}\,.$

Taking into account inequality (3), we obtain

$$l < cd^{\frac{3}{2}}d^{\frac{3}{2}}|z_0|^{\frac{1}{4}}\sigma_1^{\frac{1}{2}}\sigma_2 = cd^3|z_0|^{\frac{1}{4}}\sigma_1^{\frac{1}{2}}\sigma_2 \ (4.2)$$

Repeating the previous reasoning, we have

$$l < cd^3|z_0|^{\left(\frac{1}{2}\right)^3}\sigma_1^{\left(\frac{1}{2}\right)^2}\sigma_2^{\frac{1}{2}}\sigma_3 \ (4.3)$$

. . . .

$$l < cd^3|z_0|^{\left(\frac{1}{2}\right)^{n-1}} \sigma_1^{\left(\frac{1}{2}\right)^{n-2}} \ldots \sigma_{n-2}^{\frac{1}{2}} \sigma_{n-1} \, (4.n-1)$$

$$l < cd^3|z_0|^{\left(\frac{1}{2}\right)^{n}} \sigma_1^{\left(\frac{1}{2}\right)^{n-1}} \ldots \sigma_{n-2}^{\left(\frac{1}{2}\right)^{2}} \sigma_{n-1}^{\frac{1}{2}} \sigma_n \, (4.n)$$

$$\text{where } \sigma_i = \sqrt{\int_{t_{i-1}}^{t_i} v^2(r)dr}\,.$$

From inequality (4.n) we obtain

$$l^2 < c^2 d^6 |z_0|^{\left(\frac{1}{2}\right)^{n-1}} \sigma_1^{\left(\frac{1}{2}\right)^{n-2}} \ldots \sigma_{n-2}^{\frac{1}{2}} \sigma_{n-1} \sigma_n^2 \, (5)$$

Let's multiply the inequalities (5) and (4.n-1)

$$l^3 < c^3 d^9 |z_0|^{\left(\frac{1}{2}\right)^{n-2}} \sigma_1^{\left(\frac{1}{2}\right)^{n-3}} \ldots \sigma_{n-2} \sigma_{n-1}^2 \sigma_n^2 \, (6)$$

Continuing our calculations, we obtain the inequalities

$$l^{n+1} < c^{n+1} d^{3(n+1)} |z_0| \sigma_1^2 \ldots \sigma_{n-1}^2 \sigma_n^2 \, (7)$$

That's why

$$l < cd^3 (|z_0| \sigma_1^2 \ldots \sigma_{n-1}^2 \sigma_n^2)^{\frac{1}{n+1}} \leq$$

$$cd^3 \cdot \frac{|z_0| + \sigma_1^2 + \ldots + \sigma_{n-1}^2 + \sigma_n^2}{n+1} \leq \frac{cd^3(|z_0| + \sigma^2)}{n+1},$$

where

$$\sigma_1^2 + \sigma_2^2 + \ \ldots + \sigma_n^2 = \int_0^{t_1} v^2(r)dr + \int_{t_1}^{t_2} v^2(r)dr + \cdots + \int_{t_{n-1}}^{t_n} v^2(r)dr =$$

$$= \int_0^{t_n} v^2(r)dr \leq \int_0^{\infty} v^2(r)dr \leq \sigma^2.$$

Therefore

$$n + 1 < \frac{cd^3(|z_0| + \sigma^2)}{l},$$

or

$$n \leq \left[\frac{cd^3(|z_0| + \sigma^2)}{l}\right] - 1 = N - 1.$$

Therefore, for some $k, 1 \leq k \leq N, z_k = z(t_k) \in M$ и

$$t_k = d|z_0| + d|z_1| + \cdots + d|z_{k-1}| \ (8)$$

Let's estimate $|z_i|, i = 2, 3, \dots, k-1$, and get

$$|z_2| = \left|\int_{t_1}^{t_2} e^{(t_2-r)C} v(r)dr\right| \leq d\sqrt{t_2 - t_1} \cdot \sigma_2 = d\sqrt{d|z_1|} \cdot \sigma_2 =$$

$$= d^{\frac{3}{2}}|z_1|^{\frac{1}{2}}\sigma_2 \leq d^{\frac{3}{2}}d^{\frac{3}{4}}|z_0|^{\frac{1}{4}}\sigma_1^{\frac{1}{2}}\sigma_2 \leq d^3K\theta^2,$$

$$|z_3| \leq d^3K\theta^2,$$

. . . .

$$|z_{k-1}| \leq d^3K\theta^2.$$

From (8) it follows

$$t_k = d|z_0| + d^4K\theta^2(k-1) \leq d|z_0| + d^4K\theta^2(N-1) = T(z_0).$$

Hence, in the game (1) the pursuit ends from the initial position z_0 during $T(z_0)$.

The theorem has been proven.

Example. Let the differential game be described by the equation

$$\dot{z} = -u + v, (9)$$

where $z \in R^n, |u| \leq 1, \|v(\cdot)\| \leq \sigma. M = \{z: |\pi z| < l\}$.

It's clear that $\|e^{tC}\| = \|E\| = 1 = d$. Hence, the condition of the theorem is satisfied, and in the game (9) it is possible to complete the pursuit from an arbitrary point $z_0 \notin M$ during

$$T(z_0) = |z_0| + K\theta^2(N-1),$$

where $K = \max(|z_0|, 1),\ \theta = \max(\sigma, 1), N = \left[\frac{|z_0|+\sigma^2}{l}\right]$.

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